

# Crossing Number is Hard for Kernelization

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## Abstract

The graph crossing number problem,  $\text{cr}(G) \leq k$ , asks for a drawing of a graph  $G$  in the plane with at most  $k$  edge crossings. Although this problem is in general notoriously difficult, it is fixed-parameter tractable for the parameter  $k$  [Grohe]. This suggests a closely related question of whether this problem has a *polynomial kernel*, meaning whether every instance of  $\text{cr}(G) \leq k$  can be in polynomial time reduced to an equivalent instance of size polynomial in  $k$  (and independent of  $|G|$ ). We answer this question in the negative. Along the proof we show that the tile crossing number problem of twisted planar tiles is NP-hard, which has been an open problem for some time, too, and then employ the complexity technique of cross-composition. Our result holds already for the special case of graphs obtained from planar graphs by adding one edge.

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## 1 Introduction

We refer to Sections 2,3 for detailed formal definitions. Briefly, the *crossing number*  $\text{cr}(G)$  of a graph  $G$  is the minimum number of pairwise edge crossings in a drawing of  $G$  in the plane. Finding the crossing number of a graph is one of the most prominent hard optimization problems in geometric graph theory [10] and is NP-hard already in very restricted cases, e.g., for cubic graphs [12], and for graphs with prescribed edge rotations [16]. Concerning approximations, there exists  $c > 1$  such that the crossing number cannot be approximated within the factor  $c$  in polynomial time [5]. Moreover, the following very special case of the problem is still hard – a result that greatly inspired our paper:

► **Theorem 1** (Cabello and Mohar [6]). *Let  $G$  be an almost-planar graph, i.e.,  $G$  having an edge  $e \in E(G)$  such that  $G \setminus e$  is planar (called also near-planar in [6]). Let  $k \geq 1$  be an integer. Then it is NP-complete to decide whether  $\text{cr}(G) \leq k$ .*

On the other hand, it has been shown that the problem is *fixed-parameter tractable* when parameterized by itself: one can decide whether  $\text{cr}(G) \leq k$  in quadratic (Grohe [11]) and even linear (Kawarabayashi–Reed [13]) time while having  $k$  fixed. Fixed-parameter tractability (FPT) is closely related to the concept of so called *kernelization*. In fact, one can easily show that a (decidable) problem  $\mathcal{A}$  parameterized by an integer  $k$  is FPT if, and only if, every instance of  $\mathcal{A}$  can be in polynomial time reduced to an equivalent instance (the *kernel*) of size bounded only by some function of  $k$ . This function of  $k$ , bounding the kernel size, may in general be arbitrarily huge. Though, the really interesting case is when the kernel size may be bounded by a polynomial function of  $k$  (a *polynomial kernel*).

The nature of the methods used in [11, 13], together with the recent great advances in algorithmic graph minors theory, might suggest that the crossing number problem  $\text{cr}(G) \leq k$



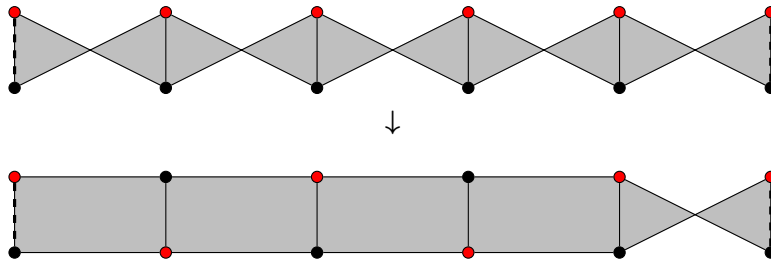
should have a polynomial kernel in  $k$ , as many related FPT problems do. This question was raised as open, e.g., at WorKer 2015 [unpublished]. Polynomial kernels for some special crossing number problem instances were constructed before, e.g., in [1]. The general result is, however, very unlikely to hold as our main result claims:

► **Theorem 2.** *Let  $G$  be an almost-planar graph, i.e.,  $G$  having an edge  $e \in E(G)$  such that  $G \setminus e$  is planar. Let  $k \geq 1$  be an integer. The crossing number problem, asking if  $\text{cr}(G) \leq k$  while parameterized by  $k$ , does not admit a polynomial kernel unless  $\text{NP} \subseteq \text{coNP}/\text{poly}$ .*

In order to prove Theorem 2, we use the technique of *cross-composition* [2]. While its formal description is postponed till Section 3, here we very informally outline the underlying idea of cross-composition. Imagine we have an NP-hard language  $\mathcal{L}$  such that we can “OR-cross-compose” an arbitrary collection of instances  $x_1, x_2, \dots, x_t$  of  $\mathcal{L}$  into the crossing number problem  $\text{cr}(G_0) \leq k_0$  for suitable  $G_0$  and  $k_0$  efficiently depending on  $x_1, x_2, \dots, x_t$ . By the words “OR-cross-compose” we mean that  $\text{cr}(G_0) \leq k_0$  holds if and only if  $x_i \in \mathcal{L}$  for some  $1 \leq i \leq t$  (informally,  $x_1 \in \mathcal{L}$  OR  $x_2 \in \mathcal{L}$  OR  $\dots$ ). Now assume we could always reduce a crossing number instance  $\langle G, k \rangle$  into an equivalent instance of size  $p(k)$  where  $p$  is a polynomial. Then, for the instance  $\langle G_0, k_0 \rangle$  and suitable  $t$  such that  $p(k_0) \ll t \approx |G_0| \ll 2^{|x_i|}$ , such a reduction effectively means that we should somehow decide many of the  $t$  instances  $x_i \in \mathcal{L}$  in time polynomial in  $|G_0|$  (which is  $\ll 2^{|x_i|}$ ). The latter sounds highly unlikely [9] in the complexity theory.

The task is to find a suitable NP-hard language  $\mathcal{L}$  for the aforementioned construction. While the ordinary crossing number problem is not suitable for cross-composition (roughly, since the crossing numbers of disjoint instances sum up together), a helping hand is given by the concept of the *tile crossing number* [17], defined in detail in Section 2.

Informally, a *tile* is a graph  $T$  with two disjoint sequences of vertices defining the left and right walls of  $T$ . A *tile drawing* is a drawing of  $T$  inside a rectangle such that the walls of  $T$  lie respectively on the left and right sides of this rectangle. A tile  $T$  is planar if  $T$  admits a tile drawing without crossings, and  $T$  is *twisted planar* if  $T$  becomes a planar tile after inverting (upside-down) one of the walls. As observed by Schaefer [20], the tile crossing number problem is NP-hard by a trivial reduction from ordinary crossing number, but we need much more. In order to embed the tile crossing number problem in a cross-composition construction, which will be realized as a concatenation of the tile instances across their respective walls, we shall use only twisted planar tiles. See Figure 1. The underlying idea which makes the cross-composition work, is that only one of the tile instances is drawn twisted in the concatenation and all the other contribute no crossings.



■ **Figure 1** Schematic concatenation of an odd number of twisted planar tiles; in fact, only one (and an arbitrary one) of the tiles needs to be drawn twisted in this case.

Hence the proof of Theorem 2 would be finished, modulo technical details, if we show that the tile crossing number problem of twisted planar tiles is NP-hard. This particular question seems to have been latently considered in the crossing number community for the past several years, and it is still open nowadays to our best knowledge. We provide the following affirmative answer by adapting a construction from the proof [6] of Theorem 1:

► **Theorem 3** (Corollary 12). *Let  $T$  be a twisted planar tile and  $k \geq 1$  an integer. Then it is NP-complete to decide whether there exists a tile drawing of  $T$  with at most  $k$  edge crossings. Furthermore, the same holds if both the walls of  $T$  are of size two and there exists an edge  $e \in E(T)$  such that  $T \setminus e$  is a planar tile.*

**Paper organization.** We provide the necessary formal definitions of the aforementioned concepts from crossing numbers and parameterized complexity in Sections 2,3. Then we prove Theorem 3 in Section 4, and provide technical claims useful for the next cross-composition construction in Section 5. Finally, we summarize the paper and present some additional ideas in Section 6.

## 2 Crossing numbers

We consider multigraphs by default, even though we could always subdivide parallel edges in order to make the graphs simple. We follow basic terminology of topological graph theory, see e.g. [15]. A *drawing* of a graph  $G$  in the plane is such that, the vertices of  $G$  are distinct points and the edges are simple curves joining their endvertices. It is required that no edge passes through a vertex, and no three edges cross in a common point.

► **Definition 4** (crossing number). The *crossing number*  $\text{cr}(G)$  of a graph  $G$  is the minimum number of crossing points of edges in a drawing of  $G$  in the plane.

Hence, a graph  $G$  is planar if and only if  $\text{cr}(G) = 0$ . Note that the crossing number is invariant under subdividing edges of  $G$ .

A useful concept in crossing numbers research are tiles. They were used already by Kochol [14] and Richter–Thomassen [19], although they were formalized only later in the work of Pinnontoan and Richter [17, 18]. So far, primary use of the tile concept in crossing numbers research concerned study of so called crossing-critical graphs, as can be seen also in recent papers such as [3, 4]. Here we will use tiles in a rather different way. We briefly sketch the necessary terms as follows.

A *tile* is a triple  $T = (G, \lambda, \rho)$  where  $\lambda, \rho \in V(G)^*$  are two disjoint sequences of distinct vertices of  $G$ , called the *left and right wall* of  $T$ , respectively. A *tile drawing* of  $T$  is a drawing of the underlying graph  $G$  in the unit square such that the vertices of  $\lambda$  occur in this order on the left side of the square and those of  $\rho$  in this order on the right side of it. The *tile crossing number*  $\text{tcr}(T)$  of a tile  $T$  is the minimum number of crossing points of edges over all tile drawings of  $T$ . The *right-inverted* tile  $T^\uparrow$  is the tile  $(G, \lambda, \bar{\rho})$  and the *left-inverted* tile  $\downarrow T$  is  $(G, \bar{\lambda}, \rho)$ , where  $\bar{\lambda}$  and  $\bar{\rho}$  denote the inverted sequences of  $\lambda, \rho$ .

For simplicity, in this brief exposition, we shall assume that all tiles involved in one construction satisfy  $|\lambda| = |\rho| = w$  for suitable  $w \geq 2$  (though, a more general treatment is obviously possible). The *join of two tiles*  $T = (G, \lambda, \rho)$  and  $T' = (G', \lambda', \rho')$  is defined as the tile  $T \otimes T' := (G'', \lambda, \rho')$ , where  $G''$  is the graph obtained from the disjoint union of  $G$  and  $G'$ , by identifying  $\rho(i)$  with  $\lambda'(i)$  for  $i = 1, \dots, w$ . Since the operation  $\otimes$  is associative, we can safely define the join of a sequence of tiles  $\mathcal{T} = (T_1, T_2, \dots, T_m)$  as the tile given by  $\otimes \mathcal{T} = T_1 \otimes T_2 \otimes \dots \otimes T_m$ .

A tile  $T = (G, \lambda, \rho)$  is *planar* if  $\text{tcr}(T) = 0$ , and  $T$  is *twisted planar* if  $\text{tcr}(T^\dagger) = 0$  (which is clearly equivalent to  $\text{tcr}(\dagger T) = 0$ ). We briefly illustrate these definitions (also Figure 1):

► **Example 5.** Let  $\mathcal{T} = (T_1, T_2, \dots, T_m)$  be a sequence of twisted planar tiles  $T_i, i = 1, \dots, m$ . Then  $\text{tcr}(\otimes \mathcal{T}) = 0$  if  $m$  is even, and  $\text{tcr}(\otimes \mathcal{T}) \leq \min_{i \in \{1, \dots, m\}} \text{tcr}(T_i)$  otherwise.

Finally, the following is a useful artifice in crossing numbers research. In a *weighted* graph, each edge is assigned a positive number (the *weight*, or *thickness* of the edge). Now the crossing number is defined as in the ordinary case, but a crossing point between edges  $e_1$  and  $e_2$ , say of weights  $t_1$  and  $t_2$ , contributes  $t_1 \cdot t_2$  to the result. In the case of integer weights, this extension can be easily seen equivalent to the unweighted setting as follows:

► **Proposition 6** (folklore). *Let  $G$  be an integer-weighted graph,  $F \subseteq E(G)$ , and  $G^+$  be constructed from  $G$  via replacing each edge  $e \in F$  of weight  $t$  with a bunch of  $t$  parallel edges of weight 1. Then  $\text{cr}(G) = \text{cr}(G^+)$ . Moreover, if  $G$  is the graph of a tile  $T$  and  $T^+$  is the corresponding tile based on  $G^+$ , then  $\text{tcr}(T) = \text{tcr}(T^+)$ .*

### 3 Parameterized complexity and kernelization

Here we introduce the relevant concepts of parameterized complexity theory. For more details, we refer to textbooks [7, 8]. Let  $\Sigma$  be a finite alphabet. A parameterized problem over  $\Sigma$  is a language  $\mathcal{A} \subseteq \Sigma^* \times \mathbb{N}$ . An instance of  $\mathcal{A}$  is thus a pair  $\langle x, k \rangle$  where  $x$  is the input and  $k \geq 0$  an (integer) parameter. In our case, e.g.,  $\langle G, k \rangle$  is the crossing number instance “ $\text{cr}(G) \leq k$ ”. A parameterized problem is *fixed-parameter tractable* (FPT) if every instance  $\langle x, k \rangle$  can be solved in time  $f(k) \cdot |x|^c$ , where  $f$  is a computable function and  $c$  is a constant.

A hot research direction in the area of parameterized complexity of the past decade is that of kernelization. A *kernelization* for a parameterized problem  $\mathcal{A}$  is an algorithm that takes an instance  $\langle x, k \rangle$  of  $\mathcal{A}$  and, in time polynomial in  $|x| + k$ , maps  $\langle x, k \rangle$  to an equivalent instance  $\langle x', k' \rangle$  of  $\mathcal{A}$  such that  $|x'| + k' \leq f(k)$  where  $f$  is a computable function. The output  $\langle x', k' \rangle$  is called the *kernel*. We say that  $\mathcal{A}$  has a *polynomial kernel* if there is a kernelization for  $\mathcal{A}$  such that  $f$  is a polynomial. Every fixed-parameter tractable problem admits a kernel, but not necessarily a polynomial kernel.

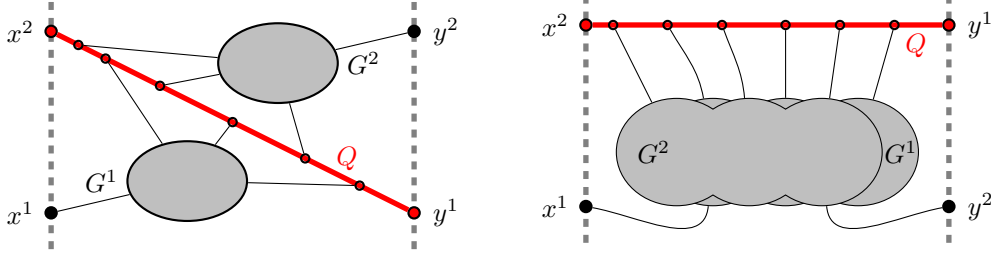
We now describe the basic OR-cross-composition framework of [2]. An equivalence relation  $\sim$  on  $\Sigma^*$  is called a polynomial equivalence if, for any  $x, y \in \Sigma^*$ , we can decide in polynomial time whether  $x \sim y$  and, moreover, on any finite  $S \subseteq \Sigma^*$  the relation  $\sim$  defines a number of equivalence classes which is polynomially bounded in the size of a largest element of  $S$ . For our purpose,  $\sim$  will group together the tile crossing number instances of the same objective value  $k$ .

► **Definition 7** (OR-cross-composition). Let  $\mathcal{L} \subseteq \Sigma^*$  be a language,  $\sim$  be a polynomial equivalence relation on  $\Sigma^*$ , and let  $\mathcal{A} \subseteq \Sigma^* \times \mathbb{N}$  be a parameterized problem. An *OR-cross-composition* of  $\mathcal{L}$  into  $\mathcal{A}$  is an algorithm that, given  $t$  instances  $x_1, x_2, \dots, x_t \in \Sigma^*$  of  $\mathcal{L}$  such that  $x_1 \sim x_2 \sim \dots \sim x_t$ ;

- in time polynomial in  $|x_1| + \dots + |x_t|$  it outputs an instance  $\langle y_0, k_0 \rangle \in \Sigma^* \times \mathbb{N}$  such that  $k_0$  is polynomially bounded in  $\max_i |x_i| + \log t$ , and
- $\langle y_0, k_0 \rangle \in \mathcal{A}$  if and only if  $x_i \in \mathcal{L}$  for some  $1 \leq i \leq t$ .

► **Theorem 8** (Bodlaender, Jansen and Kratsch [2]). *If an NP-hard language  $\mathcal{L}$  has an OR-cross-composition into the parameterized problem  $\mathcal{A}$ , then  $\mathcal{A}$  does not admit a polynomial kernel unless  $\text{NP} \subseteq \text{coNP}/\text{poly}$ .*

We remark in passing that the full claim of [2] is even stronger than stated Theorem 8, and in particular it also excludes the existence of a so-called polynomial compression of  $\mathcal{A}$ .



■ **Figure 2** A diagonally separated tile and a possible drawing of the corresponding right-inverted tile. The underlying graph of this tile contains two vertex-disjoint subgraphs  $G^1, G^2$  such that  $V(G^1) \cup V(G^2) = V(G) \setminus \{x^2, y^1\}$ , and their drawings “overlay” each other on the right.

#### 4 Twisted planar tiles

For the purpose of our proof, we are especially interested in the following kind of integer-weighted planar tiles. See Figure 2 for an illustration.

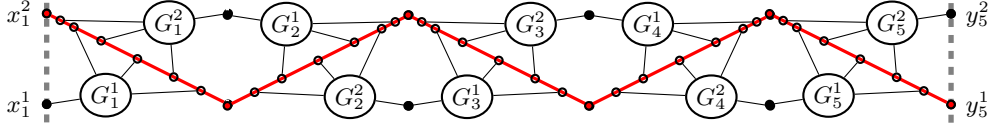
- **Definition 9** (diagonally separated tile). Consider an integer-weighted planar tile  $T = (G, \lambda, \rho)$  where the walls are  $\lambda = (x^1, x^2)$  and  $\rho = (y^1, y^2)$  for some distinct  $x^1, x^2, y^1, y^2 \in V(G)$ . We say that  $T$  is *diagonally separated* if we can write  $G = G^1 \cup G^2 \cup Q$  such that
- $G^1, G^2$  are vertex-disjoint subgraphs of  $G$  such that  $V(G^1) \cup V(G^2) = V(G) \setminus \{x^2, y^1\}$ ,
  - $E(Q) = E(G) \setminus (E(G^1) \cup E(G^2))$ ,  $x^1, y^2 \notin V(Q)$ , and  $Q$  is a “thick” path from  $x^2$  to  $y^1$  having each edge of weight  $t \geq w_1 \cdot w_2 + 1$  where  $w_i$  is the sum of weights of all the edges of the subgraph  $G^i \setminus V(Q)$ ,
  - $G$  is connected and both  $G^1 \setminus V(Q)$  and  $G^2 \setminus V(Q)$  are connected subgraphs, and no edge of  $G^1 \cup G^2$  has both ends in  $V(Q) \cup \{x^1, y^2\}$ ,
  - $x^1 \in V(G^1) \setminus V(Q)$  and  $y^2 \in V(G^2) \setminus V(Q)$ , both the vertices  $x^1, y^2$  are of degree one in  $G$  and the two incident edges have weight 1.

Twisted diagonally separated planar tiles have the suitable “or-composability” property:

- **Lemma 10.** Let  $\mathcal{T} = (T_1^\uparrow, T_2^\uparrow, \dots, T_m^\uparrow)$  be a sequence of tiles such that, for  $i = 1, \dots, m$ ,  $T_i$  is a diagonally separated planar tile. Let  $U := \otimes \mathcal{T}$  if  $m$  is odd, and  $U := (\otimes \mathcal{T})^\uparrow$  otherwise. Then  $\text{tcr}(U) = \min_{i \in \{1, \dots, m\}} \text{tcr}(T_i^\uparrow)$ .

**Proof.** Let the underlying graph of  $T_i$  be  $G_i^1 \cup G_i^2 \cup Q_i$ , as anticipated by Definition 9, and let  $t_i$  be the weight of  $E(Q_i)$ . Let  $(x_i^1, x_i^2), (y_i^2, y_i^1)$  be the left and right walls, respectively, of  $T_i^\uparrow$ . By the definition of join  $\otimes$ ,  $y_i^1 = x_{i+1}^2$  and hence  $Q := Q_1 \cup \dots \cup Q_m$  is a path from  $x_1^2$  to  $y_m^1$ . Similarly,  $y_i^2 = x_{i+1}^1$ , and so  $H_i := G_i^2 \cup G_{i+1}^1$  is a connected component of  $U \setminus V(Q)$  for  $i = 1, \dots, m-1$ . See Figure 3. For simplicity, we let  $H_0 := G_1^1$  and  $H_m := G_m^2$  which are also components of  $U \setminus V(Q)$ .

It clearly holds  $\text{tcr}(U) \leq \min_{i \in \{1, \dots, m\}} \text{tcr}(T_i^\uparrow)$ . Furthermore, we claim that  $t_i > \text{tcr}(T_i^\uparrow)$  and so  $t_i > \text{tcr}(U)$  for each  $i = 1, \dots, m$ . Since  $T_i$  is planar, each of  $G_i^1, G_i^2$  has a plane embedding in which the vertices adjacent to  $V(Q) \cup \{x_i^1, y_i^2\}$  lie on the outer face. Consequently, there is a tile drawing of  $T_i^\uparrow$  with each of  $G_i^1, G_i^2$  plane and crossings only between the edges of  $G_i^1$  and of  $G_i^2$  that are not incident to  $Q$ . See Figure 2 right. By standard arguments, we may assume that no two edges cross more than once in this drawing and so  $\text{tcr}(T_i^\uparrow) \leq w_i^1 \cdot w_i^2 \leq t_i - 1$  where  $w_i^j$  is the sum of weights of all the edges of  $G_i^j \setminus V(Q)$ .



■ **Figure 3** The planar tile  $U^\dagger$  where  $U$  for  $m = 5$  is from the statement of Lemma 10.

From  $t_i > \text{tcr}(U)$  for  $i = 1, \dots, m$  we get that no edge of  $Q$  is ever crossed in an optimal tile drawing of  $U$ . We may hence properly define, in any optimal tile drawing of  $U$  and for each subgraph  $H_i$ , whether whole  $H_i$  lies (is drawn) *above* or *below*  $Q$ . We aim to show that there always exists  $i \in \{1, \dots, m\}$  such that  $H_{i-1}, H_i$  are drawn on the same side of  $Q$ , either both above or both below  $Q$ .

Assume the contrary. Then  $H_0$  is drawn below  $Q$  by the left wall  $(x_1^1, x_1^2)$  of  $U$ . Next,  $H_1$  is drawn above,  $H_2$  below,  $\dots$ , and finally,  $H_m$  should be drawn above  $U$  if  $m$  is odd and below  $U$  otherwise. That is exactly the opposite position to what is requested by the right wall of  $U$  which is  $(y_m^2, y_m^1)$  if  $m$  is odd and  $(y_m^1, y_m^2)$  otherwise, a contradiction.

So,  $H_{i-1}$  and  $H_i$  are drawn on the same side of  $Q$  for some  $i \in \{1, \dots, m\}$ . First assume that  $1 < i < m$ . By supposed connectivity of  $G_{i-1}$  there is a path  $P^1 \subseteq G_{i-1}^2$  from  $x_i^1 = y_{i-1}^2$  to an internal vertex of  $Q_{i-1}$ , and similarly, there is a path  $P^2 \subseteq G_{i+1}^1$  from  $y_i^2 = x_{i+1}^1$  to an internal vertex of  $Q_{i+1}$  by connectivity of  $G_{i+1}$ . Let  $D$  be the drawing obtained from a considered optimal tile drawing of  $U$  restricted to  $T_i$ , by prolonging the single weight-1 edge incident with  $x_i^1$  along  $P^1$  and the single edge incident with  $y_i^2$  along  $P^2$ . Since whole  $Q$  is uncrossed, the paths  $Q_{i-1}, Q_{i+1}$  can play the role of the left and right wall of  $D$ , and hence  $D$  is a valid tile drawing of  $T_i$  having no more crossings than  $\text{tcr}(U)$ .

If  $i = 1$  or  $i = m$ , then we directly use the left or the right wall of  $U$  in the previous argument. Consequently,  $\min_{i \in \{1, \dots, m\}} \text{tcr}(T_i^\dagger) \leq \text{tcr}(U)$  and the proof is finished. ◀

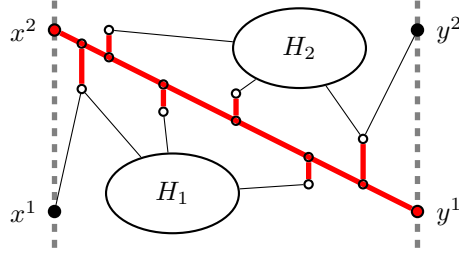
The last step of this section is to prove that the tile crossing number problem is NP-hard for twisted diagonally separated planar tiles. Due to their similarity to intermediate steps in the paper [6], it is no surprise that we can easily derive hardness using the same means; from NP-hardness of the so called anchored crossing number.

An *anchored graph* [6] is a triple  $(G, A, \sigma)$ , where  $G$  is a graph,  $A \subseteq V(G)$  are the anchor vertices and  $\sigma$  is a cyclic ordering (sequence) of  $A$ . An *anchored drawing* of  $(G, A, \sigma)$  is a drawing of  $G$  in a closed disc  $\Delta$  such that the vertices of  $A$  are placed on the boundary of  $\Delta$  in the order specified by  $\sigma$ , and the rest of the drawing lies in the interior of  $D$ . The *anchored crossing number*  $\text{acr}(G, A, \sigma)$ , or shortly  $\text{acr}(G)$ , is the minimum number of pairwise edge crossings in an anchored drawing of  $(G, A, \sigma)$ . A *planar anchored graph* is an anchored graph that has an anchored drawing without crossings. Any subgraph  $H \subseteq G$  naturally defines the corresponding anchored subgraph  $(H, A \cap V(H), \sigma|_{V(H)})$ .

► **Theorem 11** (Cabello and Mohar [6]). <sup>1</sup> *Let  $G$  be an anchored graph that can be decomposed into two vertex-disjoint connected planar anchored subgraphs. Let  $k \geq 1$  be an integer. Then it is NP-complete to decide whether  $\text{acr}(G) \leq k$ .*

<sup>1</sup> Note that [6] in general deals with weighted crossing number, in the same way as we do e.g. in Proposition 6. However, since their weights are always polynomial in the graph size, Theorem 11 holds also for unweighted graphs.





■ **Figure 4** Constructing a twisted diagonally separated planar tile; proof of Corollary 12.

► **Corollary 12.** *Let  $T$  be a diagonally separated planar tile, and  $k \geq 1$  be an integer. Then it is NP-complete to decide whether  $\text{tcr}(T^\dagger) \leq k$ .*

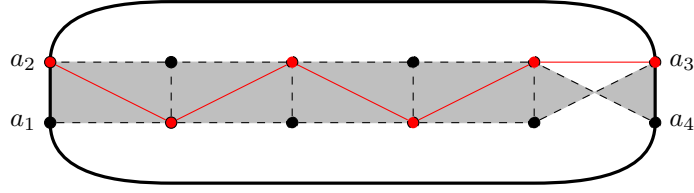
Note that twisted diagonally separated planar tiles satisfy all the assumptions of Theorem 3. In particular, if  $f$  denotes the edge incident to  $x^1$  in  $T$  then both  $T$  and  $T^\dagger \setminus f$  are planar tiles. Since the edge weights in the reduction are polynomial, the unweighted version in Theorem 3 follows immediately via Proposition 6.

**Proof.** Membership of the “ $\text{tcr}(T^\dagger) \leq k$ ” problem in NP is trivial by a folklore argument; we may guess the at most  $k$  crossings of an optimal drawing, replace those by new vertices and test planarity of the new tile. We provide the hardness reduction from Theorem 11. Let  $(G, A, \sigma)$  be an anchored graph anticipated in Theorem 11. Then  $G$  is a disjoint union of two connected components  $H_1$  and  $H_2$  where each of the corresponding anchored subgraphs  $H_1, H_2$  is a planar anchored graph. Let  $a = |\sigma|$  and  $\sigma'$  be an ordinary (non-cyclic) sequence obtained from  $\sigma$  by “opening it” at any position such that  $\sigma'(1) \in V(H_1)$  and  $\sigma'(a) \in V(H_2)$ .

Let  $Q$  be a path on the vertex set  $(x^2, s_1, s_2, \dots, s_a, y^1)$  in this order and let  $x^1, y^2$  be isolated vertices. We construct a graph  $G_0$  from the disjoint union  $G \cup Q \cup \{x^1, y^2\}$  by adding the following edges: the edges  $\{x^1, \sigma'(1)\}$  and  $\{y^2, \sigma'(a)\}$ , and the edges  $\{s_i, \sigma'(i)\}$  for  $i = 1, 2, \dots, a$ . All the edges incident with  $V(Q)$  get weight  $t = (|E(H_1)| + 1) \cdot (|E(H_2)| + 1) + 1$ , while the remaining edges have weight 1. Observe (Figure 4) that  $T_0 := (G_0, (x^1, x^2), (y^1, y^2))$  is a diagonally separated planar tile by Definition 9.

We claim that  $\text{acr}(G) \leq k$  if and only if  $\text{tcr}(T_0^\dagger) \leq k$ . In the forward direction, we take an anchored drawing of  $(G, A, \sigma)$  achieving  $\text{acr}(G)$  crossings. This drawing immediately gives (see also Figure 2 right) a tile drawing of  $T_0^\dagger$  in which “thick”  $Q$  and its incident edges, and the vertices  $x^1$  and  $y^2$ , are all drawn along the boundary of the anchored drawing without additional crossings. So, indeed,  $\text{tcr}(T_0^\dagger) \leq \text{acr}(G) \leq k$ .

In the backward direction, we observe that there is a valid tile drawing of  $T_0^\dagger$  in which the only crossings are between the edges from  $E(H_1) \cup \{x^1, \sigma'(1)\}$  and the edges from  $E(H_2) \cup \{y^2, \sigma'(a)\}$ . Consequently,  $\text{tcr}(T_0^\dagger) \leq (|E(H_1)| + 1) \cdot (|E(H_2)| + 1) = t - 1$ . Assume a tile drawing  $D_0$  (recall,  $D_0$  is contained in a unit square  $\Sigma$  with its walls on the left and right sides of  $\Sigma$ ) of  $T_0^\dagger$  with  $\text{tcr}(T_0^\dagger)$  crossings. By the previous, no “thick” edge incident with  $Q$  is crossed in  $D_0$ . Since each of the subgraphs  $H_1 + \{x^1, \sigma'(1)\}$  and  $H_2 + \{y^2, \sigma'(a)\}$  is connected, and both  $x^1, y^2$  are positioned to the same side of the ends  $x^2, y^1$  of  $Q$  on the boundary of  $\Sigma$ , both subgraphs  $H_1$  and  $H_2$  of  $G$  are drawn in the same region of  $\Sigma$  separated by the drawing of  $Q$ . Contracting the uncrossed (“thick”) edges  $\{s_i, \sigma'(i)\}$  for  $i = 1, 2, \dots, a$  hence results in an anchored drawing of  $G$  with at most  $\text{tcr}(T_0^\dagger)$  crossings. The proof is finished. ◀



■ **Figure 5** A sketch of the construction of  $G$  in the proof of Lemma 13; the cycle  $C_0$  is in bold and the tiles of  $U$  are shaded gray.

## 5 Cross-composing

We now prove the main result, Theorem 2. By Theorem 8 we know that it is enough to construct an OR-cross-composition, that is an algorithm satisfying the requirements of Definition 7.

► **Lemma 13.** *Let  $\mathcal{L}$  be the language of instances  $\langle T^\dagger, k \rangle$  where  $T$  is a diagonally separated planar tile and  $k$  an integer polynomially bounded in  $|T|$ , such that  $\text{tcr}(T^\dagger) \leq k$ . Let an equivalence relation  $\sim$  be given as  $\langle T_1^\dagger, k_1 \rangle \sim \langle T_2^\dagger, k_2 \rangle$  iff  $k_1 = k_2$ .*

*Then  $\mathcal{L}$  admits an OR-cross-composition, with respect to  $\sim$ , into the graph crossing number problem “ $\text{cr}(G) \leq k$ ” parameterized by  $k$ . Moreover, this is true even if we restrict  $G$  to be an almost-planar graph.*

**Proof.** Assume we are given  $t$  equivalent instances  $\langle T_i^\dagger, k \rangle$ ,  $i = 1, 2, \dots, t$ , of the tile crossing number problem  $\mathcal{L}$ ; “ $\text{tcr}(T_i^\dagger) \leq k$ ”. Each  $T_i$  is a diagonally separated planar tile. We construct a weighted graph  $G$  as follows (see also Figure 5 and Lemma 10):

- Let  $C_0$  be a cycle on four vertices  $a_1, a_2, a_3, a_4$  in this cyclic order, and all edges of  $C_0$  having weight  $k + 1$ .
- Let  $\mathcal{T} = (T_1^\dagger, T_2^\dagger, \dots, T_t^\dagger)$ . Let  $U := \otimes \mathcal{T}$  if  $m$  is odd, and  $U := (\otimes \mathcal{T})^\dagger$  otherwise.
- $G$  results from the union of  $C_0$  and  $U$  by identifying, in the prescribed order, the left wall of  $U$  with  $(a_1, a_2)$  and the right wall of  $U$  with  $(a_4, a_3)$ .

We show that  $\text{cr}(G) \leq k$  iff  $\text{tcr}(U) \leq k$ . In the backward direction, any tile drawing of  $U$  with  $\ell$  crossings gives a drawing of  $G$  with  $\ell$  crossings simply by embedding  $C_0$  “around” the tile  $U$ . Conversely, assume a drawing  $D$  of  $G$  with  $\ell \leq k$  crossings, and observe that no edge of  $C_0$  (weighted  $k + 1$ ) is crossed in  $D$ . Since  $G \setminus C_0$  is connected, it is drawn in one of the two faces of  $C_0$  and this clearly gives a tile drawing of  $U$  with  $\ell$  crossings.

Now, by Lemma 10,  $\text{tcr}(U) \leq k$  iff there exists  $i \in \{1, \dots, t\}$  such that  $\text{tcr}(T_i^\dagger) \leq k$ , as required by Definition 7. The construction of  $G$  is easily finished in polynomial time, and since the edge weights  $k + 1$  in  $G$  are polynomially bounded, there is a polynomial reduction to an unweighted crossing number instance by Proposition 6. It remains to verify that  $G$  is almost-planar. Let  $e_1$  be the unique edge of  $T_1$  incident with  $a_1$  in  $G$ . Then  $\text{tcr}(T_1^\dagger \setminus e_1) = \text{tcr}(T_1 \setminus e_1) = 0$  and hence  $\text{cr}(G \setminus e_1) = 0$ . ◀

Theorem 2 follows from Corollary 12 and Lemma 13 via Theorem 8 (note that  $\sim$  trivially is a polynomial equivalence).



## 6 Conclusion

We have proved that the graph crossing number problem parameterized by the number of crossings, which is known to be fixed parameter tractable, is highly unlikely to admit a polynomial kernelization. The complexity of the crossing number problem has been commonly studied under various additional restrictions on the input graph. Our negative result extends even to the instances in which the input graph  $G$  is one edge away from planarity (i.e., almost-planar  $G$ ).

On the other hand, the ordinary crossing number problem remains NP-hard for cubic graphs and for the so-called minor crossing number [12], and for graphs with a prescribed edge rotation system [16]. For a drawing of a graph, the *rotation* of a vertex is the clockwise order of its incident edges (in a local neighbourhood). A *rotation system* is the list of rotations of every vertex. As proved in [16], there is a polynomial equivalence between the problems of computing the crossing number of cubic graphs and that of computing the crossing number under prescribed rotation systems.

The construction we use to show hardness in the paper, produces instances which are “very far” from having small vertex degrees or a fixed rotation system, and there does not seem to be any easy modification for that. Nevertheless, we have an indication that the following strengthening might also be true:

► **Conjecture 14.** *Let  $G$  be a graph with a given rotation system. Let  $k \geq 1$  be an integer. The problem of whether there is a drawing of  $G$  respecting the prescribed rotation system and having at most  $k$  crossings, parameterized by  $k$ , does not admit a polynomial kernel unless  $NP \subseteq coNP/poly$ .*

*Consequently, the crossing number problem  $cr(G) \leq k$  restricted to cubic graphs  $G$ , and the analogous minor crossing number problem, do not admit a polynomial kernel w.r.t.  $k$  unless  $NP \subseteq coNP/poly$ .*

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